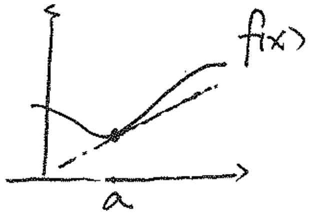
§11.10 Taylor and Maclaurin Series.

Motivation: Tangent line and Linear Approximation of $f(x)$ at $x=a$



$$f(x) \approx f(a) + f'(a)\cdot(x-a)$$

We can use a linear function of x to approximate $f(x)$ near a.

Question: How to get more precise approximation?

Answer: By higher order (degree) polynomials (power functions) of x.

Definition: Taylor Series and Maclaurin Series.

The Taylor Series of the function $f(x)$ at a (or about a) is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}\cdot(x-a)^2 + \frac{f'''(a)}{3!}\cdot(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots$$

The Maclaurin Series is the Taylor Series at 0 (i.e. $a=0$):

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\cdot(x)^n = f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}\cdot x^2 + \frac{f'''(0)}{3!}\cdot x^3 + \frac{f^{(4)}(0)}{4!}\cdot x^4 + \cdots$$

Remarks:

① Taylor and Maclaurin Series are (special) Power Series with $c_n = \frac{f^{(n)}(a)}{n!}$

② $f^{(n)}(a)$ means the nth derivative of $f(x)$ at a. eg. $f^{(4)}(2) = f''''(2)$.

☆ ③. The truncated nth degree sum (stop the sum at the nth power of x) is called $\boxed{\text{nth-degree Taylor polynomial}}$, denoted by $T_n(x)$:

☆ $$\boxed{T_n(x) = f(a) + f'(a)\cdot(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n} \quad \left(= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{n!}(x-a)^i\right)$$

④ ~~Find~~ Power Series (Maclaurin Series) in the formula sheet.

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}\cdot x^n,\ R=\infty \ ; \ \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\cdot x^n,\ R=1 \ ; \ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,\ R=1.$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n\cdot\frac{x^{2n+1}}{(2n+1)!},\ R=\infty \ ; \ \cos x = \sum_{n=0}^{\infty} (-1)^n\cdot\frac{x^{2n}}{(2n)!},\ R=\infty.$$

eg 1. Find the second-degree Taylor polynomial generated by (s16, MC) $f(x)=\frac{1}{x}$ about the points $x=3$.

Remark: Apply the formula of $T_n(x)$ at $a=3$. The key step is the $\boxed{\text{DERIVATIVE TABLE}}$

Solution: Derivative table of $f(x)=\frac{1}{x}$ at $a=3$ up to degree $n=2$.

$f \diagdown n$	$n=0$	$n=1$	$n=2$
$f^{(n)}(x)$	$f^{(0)}(x)=f(x)=\frac{1}{x}$	$f'(x)=\left(\frac{1}{x}\right)'=\frac{-1}{x^2}$	$f''(x)=\left(-\frac{1}{x^2}\right)'=(-1)\cdot(-2)\cdot\frac{1}{x^3}=\frac{2}{x^3}$
$f^{(n)}(3)$	$f(3)=\frac{1}{3}$	$f'(3)=\frac{-1}{3^2}=-\frac{1}{9}$	$f''(3)=\frac{2}{3^3}=\frac{2}{27}$

Therefore, $T_2(x)\overset{a=3}{=}f(3)+f'(3)\cdot(x-3)+\frac{f''(3)}{2!}\cdot(x-3)^2=\boxed{\frac{1}{3}-\frac{1}{9}(x-3)+\frac{2}{27}\cdot\frac{(x-3)^2}{2!}}$

eg 2. Find the 3rd degree Maclaurin polynomial of $f(x)=e^{-5x}$

Solution: Direct application of the formula for $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

$$e^{-5x}=\sum_{n=0}^{\infty}\frac{(-5x)^n}{n!}=\sum_{n=0}^{\infty}\frac{(-5)^n}{n!}\cdot x^n=1-5x+\frac{(-5x)^2}{2!}+\frac{(-5x)^3}{3!}+\cdots$$

Therefore, $\boxed{T_3(x)=1-5x+\frac{25}{2}\cdot x^2-\frac{5^3}{3!}\cdot x^3}$, and the nth term is $\boxed{\frac{(-5)^n}{n!}\cdot x^n}$.

Remark: These formulas only apply to Maclaurin Series. For Taylor Series, you still need to set the Derivative Table.

eg 2'. Find the 3rd degree Taylor polynomial of $f(x)=e^{-5x}$ at $a=-1$.

$f \diagdown n$	$n=0$	$n=1$	$n=2$	$n=3$
$f^{(n)}(x)$	e^{-5x}	$(-5)e^{-5x}$	$(-5)^2\cdot e^{-5x}$	$(-5)^3\cdot e^{-5x}$
$f^{(n)}(-1)$	e^5	$-5\cdot e^5$	$25\cdot e^5$	$-125\,e^5$

Therefore, $\boxed{T_3(x)=e^5-5e^5\cdot(x+1)+\frac{25\cdot e^5}{2!}\cdot(x+1)^2-\frac{125\cdot e^5}{3!}\cdot(x+1)^3}$

Remark: The Derivative Table method also works for eg 2. Maclaurin Series with $a=0$.

eg.3. Find the Taylor Series of $f(x)=x^3+x-5$ at $x=2$.

Remark: Taylor Series for any polynomial f will become a finite sum of the same degree.

Solution:

n	$f^{(n)}(x)$	$f^{(n)}(2)$
$n=0$	$f(x)=x^3+x-5$	$f(2)=8+2-5=5$
$n=1$	$f'(x)=3x^2+1$	$f'(2)=3\cdot4+1=13$
$n=2$	$f''(x)=6x$	$f''(2)=6\cdot2=12$
$n=3$	$f'''(x)=6$	$f'''(2)=6$
$n=4$	0	0
$n\geq4$.	0	0

Therefore,

$$f(x)=(T_3(x))=f(2)+f'(2)(x-2)+\frac{f''(2)}{2}(x-2)^2+\frac{f'''(2)}{3!}(x-2)^3$$

$$=\boxed{5+13\cdot(x-2)+\frac{12}{2}\cdot(x-2)^2+\frac{6}{3!}\cdot(x-2)^3}$$

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eg.4. Find the first three non-zero terms of the Maclaurin Series for $f(x)=e^{2x}\cdot\cos x$.

(S16, 20pts)

Solution 1: $$e^{2x}=\sum_{n=0}^{\infty}\frac{(2x)^n}{n!}=1+2x+\frac{4x^2}{2}+\cdots$$

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2}+\frac{x^4}{4!}+\cdots$$

Then $e^{2x}\cdot\cos x=(1+2x+2x^2+\cdots)\cdot(1-\frac{1}{2}x^2+\frac{1}{4!}\cdot x^4+\cdots)$

$$=(1-\tfrac{1}{2}\cdot x^2+\tfrac{1}{4!}\cdot x^4)+(2x-2x\cdot\tfrac{1}{2}\cdot x^2+2x\cdot\tfrac{1}{4!}\cdot x^4+\cdots)+(2x^2-2x^2\cdot\tfrac{1}{2}x^2+2x^2\cdot\tfrac{1}{4!}x^4+\cdots)$$

(we only need to pick all the x, x^2, x^3 terms)

$$=1-\tfrac{1}{2}\cdot x^2+2x-x^3+2x^2+\cdots$$

$$=\boxed{1+2x+\tfrac{3}{2}\cdot x^2}-x^3+\cdots$$

So the first three non-zero terms of the Maclaurin Series is $\boxed{1+2x+\frac{3}{2}x^2}$

Solution 2: Derivative Table

$n \backslash f$	$f^{(n)}(x)$	$f^{(n)}(0)$
$n=0$	$e^{2x}\cdot\cos x$.	$e^0\cdot\cos0=1$
$n=1$	$2e^{2x}\cdot\cos x-e^{2x}\cdot\sin x$	$2e^0\cdot\cos0-0=2$
$n=2$	$4e^{2x}\cos x-2e^{2x}\sin x-2e^{2x}\sin x-e^{2x}\cos x$. $=3e^{2x}\cdot\cos x-4e^{2x}\sin x$	$3e^0\cos0-4\cdot0=3$.

Therefore, $T_2(x)=$

$$f(0)+f'(0)\cdot x+\frac{f''(0)}{2!}x^2$$

$$=\boxed{1+2\cdot x+\frac{3}{2!}\cdot x^2}$$

eg5. Write the Taylor Series centered at $x=0$ for $f(x)=x\cdot\cos x$ in $\sum$-notation.

(Final 14). Solution: $f(x) = x\cdot\cos x = x\cdot \sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} x\cdot(-1)^n\cdot\frac{x^{2n}}{(2n)!} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\cdot x^{2n+1}}$

eg6. Find the Maclaurin Series for $f(x)=x\cdot\cos\sqrt{x}$.

(f15) Solution: $\cos\sqrt{x} = \cos x^{\frac{1}{2}} = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{\left(x^{\frac{1}{2}}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^n}{(2n)!}$

Therefore, $f(x)=x\cdot\cos\sqrt{x} = x\cdot\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^n}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{n+1}}{(2n)!}$

eg7. Find the first 3 non-zero terms of the Maclaurin Series for

$$f(x) = 2x\cdot\cos\sqrt{x} + 1 + 2x + 2x^2$$

Solution: According to eg6. $x\cdot\cos\sqrt{x} = (-1)^0\cdot\frac{x^{0+1}}{0!} + (-1)^1\cdot\frac{x^{1+1}}{2!} + (-1)^2\cdot\frac{x^{2+1}}{4!} + \cdots$

$= x - \frac{1}{2}\cdot x^2 + \frac{x^3}{4!} + \cdots$

Therefore, $2x\cdot\cos\sqrt{x} + 1 + 2x + x^2$

$= 2\left(x - \frac{1}{2}x^2 + \frac{x^3}{4!} + \cdots\right) + 1 + 2x + 2x^2$

$= 2x - x^2 + \frac{2x^3}{4!} + 1 + 2x + 2x^2 + \cdots = \boxed{1 + 4x + x^2} + \frac{2}{4!}x^3 + \cdots$

Hint for Webwork 6: Maclaurin Series for $f(x) = \cosh 3x$.

$f(x) = \frac{1}{2}(e^{3x} + e^{-3x}) = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(3x)^n}{n!} + \sum_{n=0}^{\infty}\frac{(-3x)^n}{n!}\right] = \sum_{n=0}^{\infty}\frac{1}{2}\cdot\frac{3^n + (-3)^n}{n!}\cdot x^n$

Notice that $n=0$, $c_0 = \frac{1}{2}\cdot\frac{1+1}{1} = 1$; $n=1$. $c_1 = \frac{1}{2}\cdot\frac{3-3}{1} = 0$

$n=2$, $c_2 = \frac{1}{2}\cdot\frac{3^2+(-3)^2}{2!} = \frac{3^2}{2!}$; $n=3$. $c_3 = \frac{1}{2}\cdot\frac{3^3-3^3}{3!} = 0$

....

Therefore, if n is odd, the corresponding coefficient is ZERO. Only even terms are left.

i.e. $f(x) = \sum_{n=0}^{\infty}\frac{3^{2n}}{(2n)!}\cdot x^{2n}$, So the nth (nonzero) term is $\frac{3^{2n}\cdot x^{2n}}{(2n)!}$

$= 1 + \frac{3^2}{2!}\cdot x^2 + \frac{3^4}{4!}x^4 + \cdots$

and $T_3(x) = 1 + 0\cdot x + \frac{3^2}{2!}\cdot x^2 + 0\cdot x^3$ (degree up to 3)

§11.10/11 Application of Taylor/Maclaurin Series

- The Maclaurin Series of $f(x)=(1+x)^k$ is called the Binomial Series.

(*) $(1+x)^k = 1 + k\cdot x + \frac{k(k-1)}{2!}\cdot x^2 + \frac{k(k-1)(k-2)}{3!}\cdot x^3 + \cdots$, where k is a constant.

eg.1. what is the coefficient of the x term of the binomial series of $(1-\frac{x}{3})^{-3}$

Remark: It is equivalent to ask you to find the coefficient in $(1-\frac{x}{3})^{-3}$'s Maclaurin Serie.

It is enough to compute $\frac{f'(0)}{1!}\cdot x$

Solution: $f'(x) = -3\cdot(1-\frac{x}{3})^{-4}\cdot(-\frac{1}{3})$, (chain rule)

$\Rightarrow f'(0) = -3\cdot(1)^{-4}\cdot(-\frac{1}{3}) = 1$. ie. the coefficient is $\boxed{1}$

Remark: In the webwork, you can use formula (*) directly.

eg 2. Find the 4th term of the binomial series $f(x)=\frac{1}{\sqrt[5]{1+x^2}}$

$f(x)=(1+x^2)^{-\frac{1}{5}}$. $k=-\frac{1}{5}$. replace x by x^2 in (*).

$= 1 + k\cdot(x^2) + \frac{k(k-1)}{2!}\cdot(x^2)^2 + \frac{k(k-1)(k-2)}{3!}\cdot(x^2)^3 + \cdots$

So the 4th term is $\frac{(-\frac{1}{5})\cdot(-\frac{1}{5}-1)(-\frac{1}{5}-2)}{3!}\cdot x^6 = \boxed{\frac{-\frac{1\times 6\times 11}{5^3}}{3!}\cdot x^6}$

- One can consider the sum, derivative, integral of Taylor Series by considering each term.

eg.3. Find the first 3 non-zero terms of $F(x)=\int_0^x t\cos t\,dt$.

According to eg.5 of §11.10. We have $F(x)=\int_0^x t\cdot\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}\,dt$

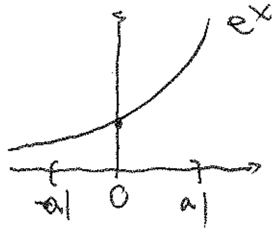
$= \int_0^x t\cdot(1-\frac{t^2}{2}+\frac{t^4}{4}+\cdots)\,dt$

$= \int_0^x (t-\frac{t^3}{2}+\frac{t^5}{4}+\cdots)\,dt = \boxed{\frac{1}{2}x^2 - \frac{1}{2}\cdot\frac{1}{4}x^4 + \frac{1}{4}\cdot\frac{1}{6}x^6} + \cdots$

- Remainder and Tail Estimate.

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. $T_n(x)$ is the nth degree Taylor polynomial. $R_n(x) = f(x) - T_n(x)$ is called the remainder and satisfies $|R_n(x)| \le \frac{M}{(n+1)!}\cdot|x-a|^{n+1}$, where $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$.

eg.4 Estimate the error in approximating e^x by $T_2(x)$ (at 0) in $x \in [-a|, a|]$



f^{n}	0	1	2	...
$f^{(n)}(x)$	e^x	e^x	e^x	...
$f^{(n)}(0)$	1	1	1	...

$T_2(x) = 1 + 1\cdot x + \frac{1}{2!}\cdot x^2$

$R_2(x) = e^x - (1 + x + \frac{1}{2}x^2)$

$|R_2(x)| \le \frac{M}{(2+1)!}\cdot|x-0|^{2+1}$, where M is the maximal value of $f^{(n+1)}(x)$ in $[-a|, a|]$

i.e. $M = \max_{x\in[-a|,a|]} e^x = e^{a|} = e^x$, $|x| \le a|$.

$\le \boxed{\frac{e^{a|}}{3!}\cdot a|^3}$

Hint for ww8: Give $|f^{(n)}(x)| \le 14$, consider $M = 14$ in Tail Estimate of $R_n(x)$.

Find the least integer n such that $\frac{14}{(n+1)!}|x|^{n+1} \le \frac{5}{10^5}$ evaluated at different x.

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eg5. Evaluate the following limit by power series: $\lim_{x\to 0} \frac{\frac{1}{1+3x^2} - 1 + 3x^2 - 9x^4}{5x^6}$

Solution: Express $\frac{1}{1+3x^2}$ as a power series:

$$= \sum_{n=0}^{\infty}(-3x^2)^n = 1 - 3x^2 + 9x^4 - 27\cdot x^6 + 81x^8 \cdots \text{(the rest terms have power 8, 10, ...)}$$

Then (*) $\frac{\frac{1}{1+3x^2} - 1 + 3x^2 - 9x^4}{5x^6} = \frac{[1 - 3x^2 + 9x^4 - 27\cdot x^6 + 81\cdot x^8 - x^{10}\cdots] - 1 + 3x^2 - 9x^4}{5x^6}$

$$= \frac{-27\cdot x^6 + 81\cdot x^8 + \cdots x^{10} + \cdots}{5x^6} = -\frac{27}{5} + \frac{81}{5}\cdot x^2 + \cdots \text{(rest terms have power higher than 2)}$$

$\boxed{\lim_{x\to 0}(*) = -\frac{27}{5}}$

Remark: Actually, we can evaluate the limit without Power Series Directly.

$$\frac{\frac{1}{1+3x^2} - 1 + 3x^2 - 9x^4}{5x^6} = \frac{\cancel{1} - \cancel{1} - \cancel{3x^2} + \cancel{3x^2} + \cancel{9x^4} - \cancel{9x^4} - 27\cdot\cancel{x^6}}{(1+3x^2)\cdot(5\cancel{x^6})} = \frac{-27}{(1+3x^2)\cdot 5} \xrightarrow{x\to 0} \boxed{\frac{-27}{5}}$$